WESLEY COLLEGE

By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER TWO 2016
TEST 4: Motion and Differential Equations

Name: ______________________

Monday 12th September Time: ~~45~~ 50 minutes Mark /~~40~~ 45

Section 1 – Calculator free 20 marks

1. [5 marks – 4 and 1]

The noise level, in decibels, of the Year 9 class next door is increasing at a rate proportional to the square root of itself, i.e. $\frac{dN}{dt} \propto \sqrt{N}$. The noise level started at 64dB and rose to 100dB in 20 minutes.

(a) Write and solve an appropriate differential equation to model this situation

$$\frac{dN}{dt} = kN^{\frac{1}{2}}$$

$$\therefore \int \frac{dN}{N^{\frac{1}{2}}} = \int k\,dt$$

$$\therefore 2\sqrt{N} = kt + c$$

$(0, 64) \Rightarrow c = 16$

$(20, 100) \Rightarrow 20 = 20k + 16$

$k = \frac{1}{5}$

ie. $2\sqrt{N} = \frac{t}{5} + 16$

$\sqrt{N} = \frac{t}{10} + 8$

$N = \left(\frac{t}{10} + 8\right)^2$

(b) How long from the start will it take to exceed the pain threshold by reaching 144 dB?

$\frac{t}{10} + 8 = 12$

$t = 40$ minutes

2. [6 marks – 1 each]

A particle is moving in simple harmonic motion with its acceleration at time t given by

$\frac{d^2x}{dt^2} = 4\cos(kt)$, for k a constant.

(a) Express each of these quantities in terms of k:

(i) the period of motion $\frac{2\pi}{k}$

(ii) the frequency of motion $\frac{k}{2\pi}$

(iii) the amplitude of the motion $\frac{4}{k^2}$

(iv) the displacement $x(t)$

$$x(t) = -\frac{4}{k^2}\cos(kt)$$

(b) If the maximum speed of the particle is 6 units, evaluate:

(v) k $\frac{4}{k} = 6 \Rightarrow k = \frac{2}{3}$

(vi) the amplitude

$$A = \frac{4}{\frac{4}{9}} = 9$$

3. [3 marks]

An object, with displacement x and velocity v, moves so that $v = 3x - 5$ m/s.

What is the acceleration of this object it is 2 metres from the origin?

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}[3x-5]^2\right)$$

$$= \frac{1}{\cancel{2}}\cdot\cancel{2}\cdot(3x-5)\cdot 3$$

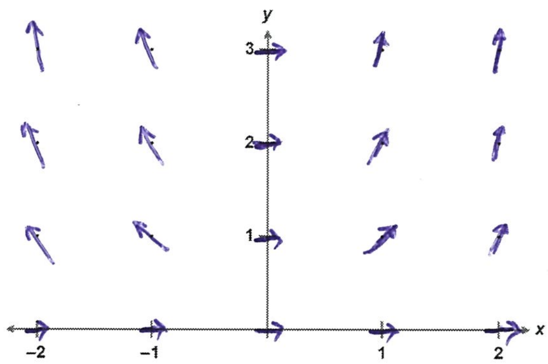
$$a(x=2) = 3 \text{ m/sec}^2$$

4. [6 marks – 1, 2 and 3]

(a) Enter the values of $\frac{dy}{dx} = xy$ in this table.

x	y	$\frac{dy}{dx}$
1	2	2
-2	3	-6
3	0	0

(b) Use these values and others from the 20 integer points marked to draw the slope field for the differential equation $\frac{dy}{dx} = xy$.



(c) Solve the differential equation $\frac{dy}{dx} = xy,\ y > 0$ if $y = 2$ when $x = 0$.

$$\int \frac{dy}{y} = \int x\, dx$$

$$\ln y = \frac{x^2}{2} + c$$

$$y = e^{\frac{x^2}{2}} \cdot C \quad \text{at } (0, 2)$$

$$= 2e^{\frac{x^2}{2}}$$

Section 2 – Calculator assumed 20 marks

Name: ______________________

5. [3 marks – 2 and 1]

(a) Continuing with $\frac{dy}{dx} = xy$, complete the table to find the coordinates of the next two points, starting from (2, 1), when the incremental formula (Euler's method) is applied.

$\delta y \approx \frac{dy}{dx} \delta x$

x	y	δx	δy
2	1	0.1	0.2
2.1	1.2	0.1	0.252
2.2	1.452	-	-

(b) What can be said about the initial (boundary) condition if $\frac{dy}{dx} = xy$ and the graph produced using Euler's method is a horizontal line?

$y = 0$.

6. [3 marks – 2 and 1]

The velocity of particle P_1 is given by $v_1(t) = 2t,\ t \geq 0$ while that of P_2 is given by $v_2(t) = t^2 + 2t - 1,\ t \geq 0$.
Both particles are moving along the same straight line and are initially at the origin O.

(a) When is the velocity of the two particles the same?

$2t = t^2 + 2t - 1$

$\Rightarrow \quad t = 1 \quad$ (only +ve sol^n)

(b) What distance is covered by P_2 up to and when the velocities are equal?

Distance $= \int_0^1 |t^2 + 2t - 1|\, dt = 0.77$ units

7. [12 marks – 2, 6, 2, 1 and 1]

An advertising executive commissioned a mathematical analysis of the effectiveness of a particular television campaign.

The rate of increase in the percentage of the market (P) aware of the product was modelled by $\frac{dP}{dt} = 2P - 0.025P^2$, at t weeks

(a) This equation has the rate of increase proportional to two basic quantities. What are they?

$$\frac{dP}{dt} = 0.025P(80-P)$$

$\propto P$, current percentage

$\propto (80-P)$ difference between a max and current P

(b) Use appropriate calculus techniques to derive $P(t) = \frac{2}{0.025 + Ce^{-2t}}$.

Isolate variables $\int \frac{dP}{P(80-P)} = \int 0.025 \, dt$

$$\int \frac{\frac{1}{80}}{P} + \frac{\frac{1}{80}}{80-P} \, dP = \int 0.025 \, dt$$

$\times 80$. $\ln P - \ln(80-P) = 2t + c$

$$\ln\left(\frac{P}{80-P}\right) = 2t + c$$

$$\frac{P}{80-P} = e^{2t+c}$$

Solve for P: $P = \frac{80e^{2t+c}}{1 + e^{2t+c}}$ $\div 40e^{2t+c}$ $\div 40e^{2t+c}$

$$= \frac{2}{0.025 + Ce^{-2t}}$$

If 20% of the market was initially aware of the product, determine:

(c) the proportion aware after 2 weeks of advertising

$$P(0) = 20 \Rightarrow \frac{2}{0.025 + C} = 20 \Rightarrow C = 0.075$$

$$\therefore \quad P(2) = \frac{2}{0.025 + 0.075e^{-4}} = 75.83\% \approx 76\%$$

or use e-act on ClassPad

(d) how long before 75% of the market is aware

$$\text{Solve} \quad \frac{2}{0.025 + 0.075e^{-2t}} = 75 \qquad t = 1.9 \text{ weeks}$$

[illegible])

(e) the maximum or limiting value of market awareness

80%

PTO for question 8

1,1

8. [7 marks – 2, 1, ~~2~~ and 2]

A steam-driven piston has a displacement, x, given by $x = 2\cos 2t - \sqrt{5}\sin 2t$

(a) Show that the piston is in simple harmonic motion.

$v = -4\sin 2t - 2\sqrt{5}\cos 2t$

$a = -8\cos 2t + 4\sqrt{5}\sin 2t$

ie $\dfrac{d^2x}{dt^2} = -4\left(2\cos 2t - \sqrt{5}\sin 2t\right) = -4x$

$= -k^2x$ for $k = 2$

Determine each of:

(b) the amplitude.

graph $x = 2\cos 2t - \sqrt{5}\sin 2t$

$\Rightarrow A = 3$ (or $2^2 + \sqrt{5}^2 = 9 = 3^2$)

(c) the first two values of t, $t \geq 0$, when the piston is in its central (mean) position

$t = 0.365$ or 1.936 units

(d) initial dirn of travel

from $x = +2$ towards $x = 0$

(negative or downwards)

(e) ~~(d)~~ the velocity when $x = \sqrt{6}$

$v^2 = k^2(A^2 - x^2)$

$= 4(9 - 6)$

$= 12$

ie, $v = \pm 2\sqrt{3}$ units